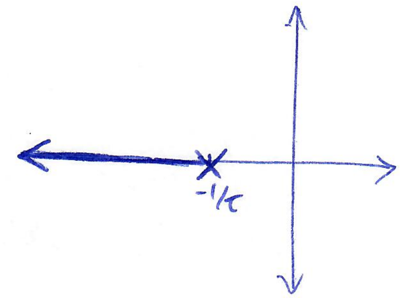
ME 4555 - Lecture 25 - Root locus design

Recall when we wanted to control a 1^{st} order plant using P and then PI control.

1) P-control: $C(s) = K$, $G(s) = \dfrac{1}{\tau s + 1}$. RL:



more K leads to faster response.

But, we need steady-state tracking! (zero error).

so we added an integrator.

2) PI-control: $C(s) = K_p + K_i \frac{1}{s}$, $G(s) = \dfrac{1}{\tau s + 1}$.

Rewrite the controller as: $C(s) = K_p \left(\dfrac{s + K_i/K_p}{s} \right)$

A root locus only allows us to change one parameter at once. So let's use $K_p = K$ as our parameter. This is equivalent to using $C(s) = K_p$ and $G(s) = \dfrac{s + K_i/K_p}{s(\tau s + 1)}$

Let's fix K_i/K_p (set the zero), and then vary K_p.

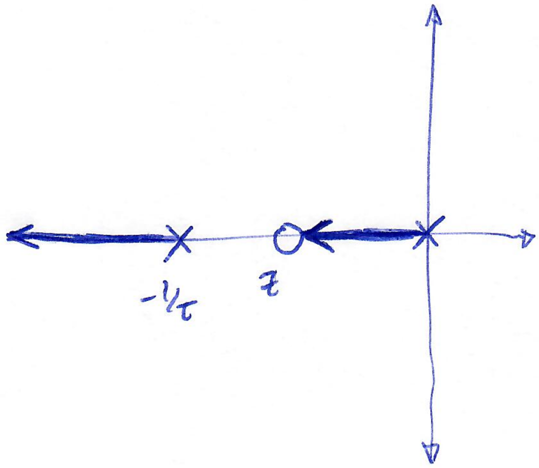
Our plant is now $\dfrac{s + \overbrace{K_i/K_p}^{-z}}{s(\tau s + 1)} \rightarrow$ poles: $\left\{ 0, -\dfrac{1}{\tau} \right\}$

zeros: $\left\{ -K_i/K_p \right\}$.

Once we fix zero $z = -K_i/K_p$

and pick gain K from RL, we get $\begin{cases} K_p = K \\ K_i = -Kz. \end{cases}$

If we place the zero to the right of $-1/\tau$, we get:



so one pole goes to z and the other goes to ∞.

Initially, this may seem bad because the zero limits how far left our dominant pole can go. (i.e. limiting settling time)

However, the closer our pole gets to z, the closer it comes to canceling it! To see why, let's try $\tau = 1$, $z = -\frac{1}{2}$.

$$G_{\not{L}}(s) = \frac{s + \frac{1}{2}}{s(s+1)} . \qquad G_{\not{c}}(s) = \frac{k(s+\frac{1}{2})}{s^2 + (k+1)s + \frac{k}{2}}$$

poles are $s = -\frac{1}{2} - \frac{k \pm \sqrt{k^2+1}}{2}$. So we can write:

$$G_{\not{c}}(s) = \frac{k(s + 1/2)}{\left(s + \frac{1}{2} + \frac{k+\sqrt{k^2+1}}{2}\right)\left(s + \frac{1}{2} + \frac{k - \sqrt{k^2+1}}{2}\right)} .$$

As $k \to \infty$, $\frac{k - \sqrt{k^2+1}}{2} \to 0$ so the $s + \frac{1}{2}$ factor cancels from num + den.

Alternatively, we can write partial fractions:

$$G_{\not{c}}(s) = \frac{\overbrace{\frac{k(k+\sqrt{k^2+1})}{\sqrt{k^2+1}(k+1+\sqrt{k^2+1})}}^{\approx 1 - \frac{1}{2k}}}{\underbrace{\frac{2}{1+k+\sqrt{k^2+1}}}_{\approx \frac{1}{k}} s + 1} - \frac{\overbrace{\frac{k(k-\sqrt{k^2+1})}{\sqrt{k^2+1}(1+k-\sqrt{k^2+1})}}^{\approx -\frac{1}{2k}}}{\underbrace{\frac{2}{1+k-\sqrt{k^2+1}}}_{\approx 2} s + 1}$$

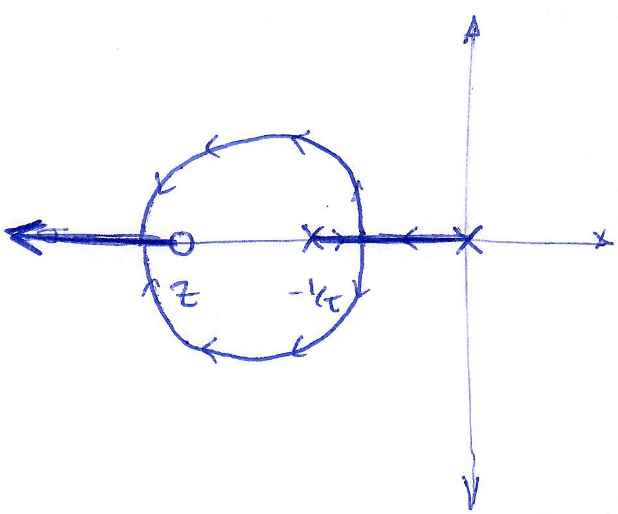
So as $k \to \infty$, $G_{\not{c}}(s) \approx \underbrace{\frac{1}{\frac{1}{k}s + 1}}_{\text{FAST}} + \underbrace{\frac{1}{2k}}_{\text{goes to zero}} \cdot \underbrace{\frac{1}{2s+1}}_{\text{SLOW}}$ goes away as $k \to \infty$ (even though it is dominant!)

Moral of the story; we saw how zeros can make transient responses worse by adding overshoot. But they can also make them better by cancelling out bad poles!

☆ test this on the interactive 2nd order response app!

> Pro tip: two wrongs don't make a right. Never get rid of unstable poles by using unstable zeros to cancel them! it's extremely unstable (because it needs to be perfect to work, and nothing ever is), will likely make things worse.

We can also add our zero to the left of $-1/\tau$, we get:



In this scenario, for some values of K, we can get oscillations as the poles move away from the real axis.

Again, as $k \to \infty$, the system behaves like it only has the fast pole, because the slow pole gets canceled by the zero.

☆ use rltool in Matlab to show how to place poles/zeros and view step response of system.

☆ also: rlocus command to draw root locus.

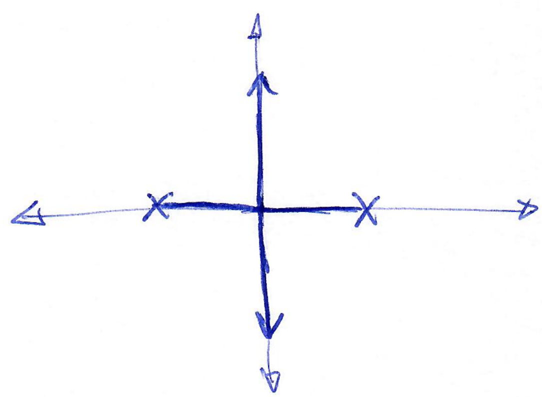
Next example: PD control of inverted pendulum.

we had $G(s) = \frac{1}{s^2 - 1}$ and $C(s) = K_p + K_d s$.

As before, let's write $C(s) = K(s - z)$. We will fix z and vary K.

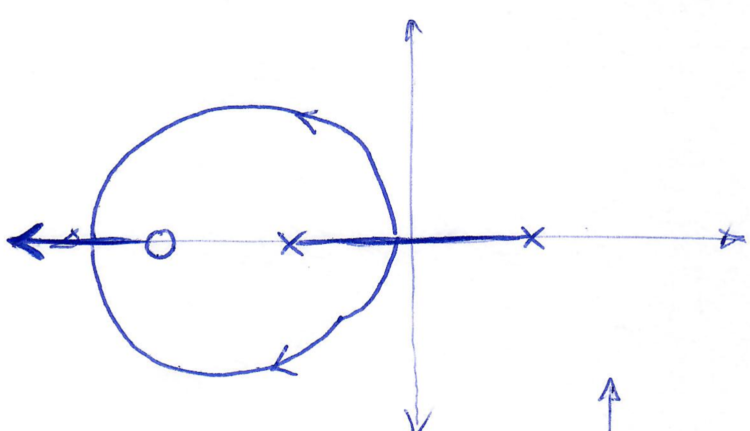
So $K_d = K$ and $K_p = -Kz$.

using P-control $(C(s) = K)$



with K large enough, we get pure oscillations (no damping, unfortunately).

using PD control $(C(s) = K(s - z))$



Same as the 1st order system case!

OR: